1.2 Length and Angle: The Dot Product

Length, distance and angle in terms of vectors

Consider the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \text{ in } \mathbb{R}^3.$$

The dot product of $\vec{u}$ and $\vec{v}$, "$\vec{u} \cdot \vec{v}$" is given by

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = 1(3) + 2(-2) + 3(4)$$

$$= 3 - 4 + 12 = 11.$$

In general, for vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^n$

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Def.

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} =$$

$$= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Clearly, ① $\vec{v}\cdot\vec{u} = \vec{u}\cdot\vec{v}$ Commutativity

We can also show easily using real numbers properties:

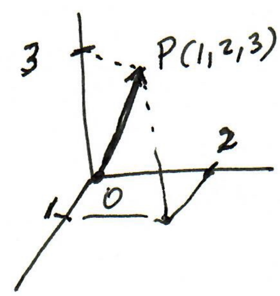
② $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$ Distributivity

③ $(c\vec{u})\cdot\vec{v} = c\,(\vec{u}\cdot\vec{v})$

④ $\vec{u}\cdot\vec{u} \geq 0$ and $\vec{u}\cdot\vec{u} = 0$ if and only if $\vec{u} = 0$.

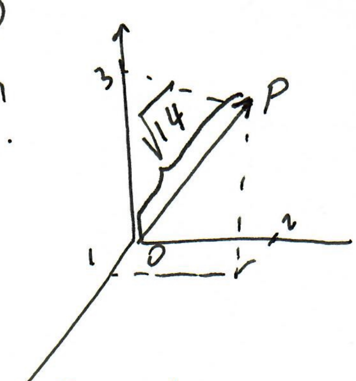
LENGTH

Consider $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$



Length of $\vec{u} = \|\vec{u}\| =$ distance from O to P(1,2,3)

$\overset{\text{Pythagoras thm}}{=} \sqrt{1^2+2^2+3^2} = \sqrt{14}$.



Def. If $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ in $\mathbb{R}^n$, then

$$\boxed{\|\vec{u}\| = \sqrt{u_1^2+\cdots+u_n^2} = \sqrt{\vec{u}\cdot\vec{u}}}$$

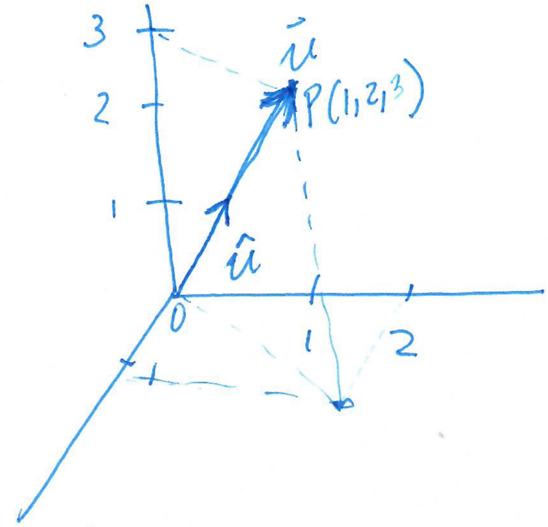
or $\boxed{\|\vec{u}\|^2 = \vec{u}\cdot\vec{u}}$

Important property

$\|c\vec{u}\| = |c|\,\|\vec{u}\|$.

Unit Vector

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$



$\|\vec{u}\| = \sqrt{14} \simeq 3.74$

We want to obtain

$\hat{u}$ in the same direction of $\vec{u}$

with $\underline{\|\hat{u}\| = 1}$

Such $\hat{u}$ can be easily obtained as

$$\hat{u} = \left(\frac{1}{\sqrt{14}}\right) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

In fact,

$$\|\hat{u}\| = \left\| \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\| = \frac{1}{\sqrt{14}} \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\| = \frac{1}{\sqrt{14}} \sqrt{1+4+9} = \frac{\sqrt{14}}{\sqrt{14}} = 1$$

<u>Definition</u>: If $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ in $\mathbb{R}^n$, then a unit vector $\hat{u}$

in the direction of $\vec{u}$ is given by

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|}$$

Two important theorems involving $\|\cdot\|$.

Theorem: Cauchy-Schwarz Inequality

For all vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^n$

$$|\vec{u}\cdot\vec{v}| \le \|\vec{u}\|\,\|\vec{v}\|.$$

Theorem Triangle Inequality

For all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$

$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|.$$

Proof. It's based on Cauchy-Schwarz.

$$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2(\vec{u}\cdot\vec{v})$$

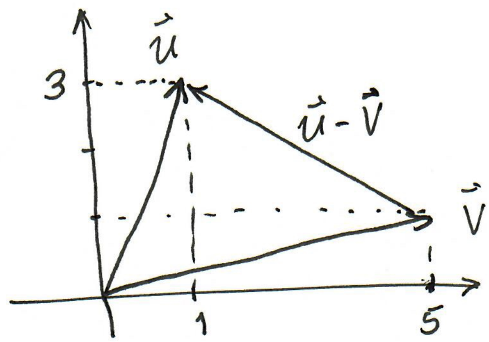
$$\le \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2|\vec{u}\cdot\vec{v}|$$

$$\overset{C\text{-}S}{\le} \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\|\vec{u}\|\,\|\vec{v}\| = \left(\|\vec{u}\| + \|\vec{v}\|\right)^2$$

$$\overset{\sqrt{}}{\Longrightarrow} \boxed{\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|.}$$

Consider the vectors:

$$\vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$



The length of the vector $\vec{u}-\vec{v}$, or

$$\|\vec{u}-\vec{v}\| = \left\| \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 5 \\ 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -4 \\ 2 \end{bmatrix} \right\|$$

$$= \sqrt{16+4} = \sqrt{20}.$$

is the distance between $\vec{u}$ and $\vec{v}$.

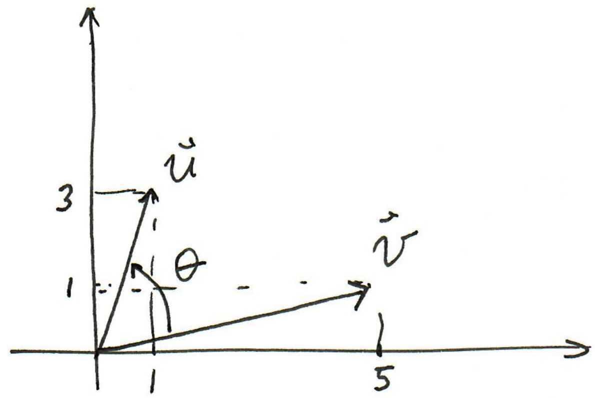
Def. Given $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

then the distance between $\vec{u}$ and $\vec{v}$ is given by

$$\|\vec{u}-\vec{v}\| = \sqrt{(u_1-v_1)^2 + \cdots + (u_n-v_n)^2}.$$

Consider again the vectors

$\vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$



How do we compute the angle θ between $\vec{u}$ and $\vec{v}$?

Formula: $\boxed{\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta}$ (5.1)

Then, if $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$

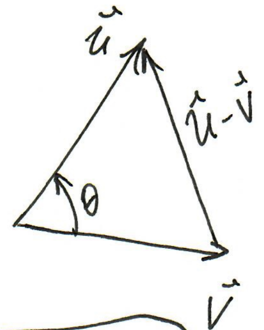
$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} = \frac{5+3}{\sqrt{10}\,\sqrt{26}} = \frac{8}{\sqrt{260}} \simeq 0.8794$$

$\Rightarrow \quad \theta = \arccos(0.8794) \simeq 1.0517$

In degrees $\simeq 60.26°$.

Formula (5.1) can be obtained as follows:

Consider the <u>law of cosines</u> (only valid in $\mathbb{R}^2, \mathbb{R}^3$)



$$\boxed{\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta} \qquad (6.1)$$

Now, we know that the left hand side

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v}).(\vec{u}-\vec{v}) =$$

$$(\vec{u}\cdot\vec{u}) - (\vec{v}\cdot\vec{u}) - (\vec{u}\cdot\vec{v}) + (\vec{v}\cdot\vec{v})$$

$$= \|\vec{u}\|^2 - 2(\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2$$

Therefore, substituting this expression into left hand side of (6.1)

$$\cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} - \cancel{2}(\vec{u}\cdot\vec{v}) = \cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} - \cancel{2}\|\vec{u}\|\|\vec{v}\|\cos\theta$$

or $$\boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta} \qquad (6.2)$$

In general,

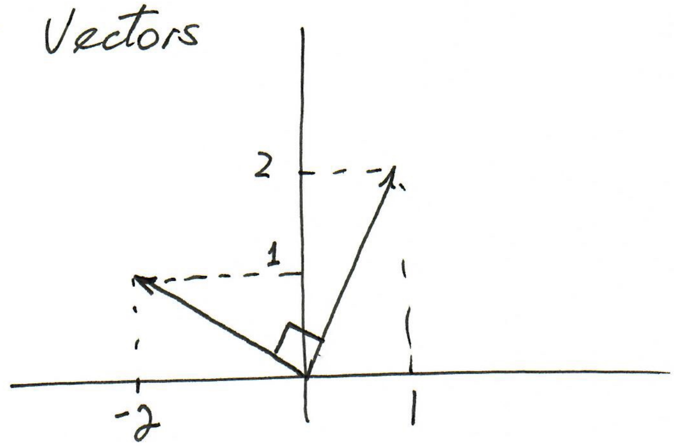
Def. If $\vec{u}, \vec{v} \neq \vec{0}$ in $\mathbb{R}^n$, then

$$\boxed{\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}} \qquad (7.1).$$

Consider the vectors

$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$



$$\vec{u} \cdot \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -2 + 2 = 0$$

Another way to look at this is using formula (6.2)

If we know $\theta = 90^\circ = \pi/2$.

Then, $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \underbrace{\cos \pi/2}_{=0} = 0.$

This is equivalent to say:

the angle between $\vec{u} \neq 0$ and $\vec{v} \neq 0$ is $\pi/2$

if and only if $\vec{u} \cdot \vec{v} = 0$

Using formula (7.1)

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{0}{\sqrt{5}\sqrt{5}} = \frac{0}{5} = 0.$$

$$\Rightarrow \quad \theta = \pi/2$$

Also, if $\theta = \pi/2$, $\vec{u}\neq\vec{0},\ \vec{v}\neq\vec{0}$ using formula (7.1)

$$0 = \cos \pi/2 = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} \Rightarrow \vec{u}\cdot\vec{v} = 0$$

or $\vec{u}\perp\vec{v}$.

In other words,

the angle between $\vec{u}\neq\vec{0}$, $\vec{v}\neq 0$ is $\pi/2$ (orthogonal)

if and only if

$$\vec{u}\cdot\vec{v} = 0$$

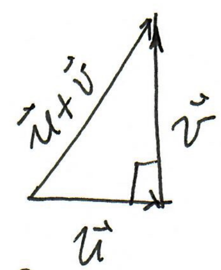
Extension to $\mathbb{R}^n$,

Def. $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ are orthogonal if and only if

$$\boxed{\vec{u} \cdot \vec{v} = 0} \qquad (8.1)$$

Pythagoras Thm

In $\mathbb{R}^2$,



$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$, this is true because the angle btw $\vec{u}$ and $\vec{v}$

In general

Thm 1.6 $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ satisfy

$$\boxed{\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2} \qquad (8.2)$$

if and only if $(\vec{u} \cdot \vec{v}) = 0$.

Proof.

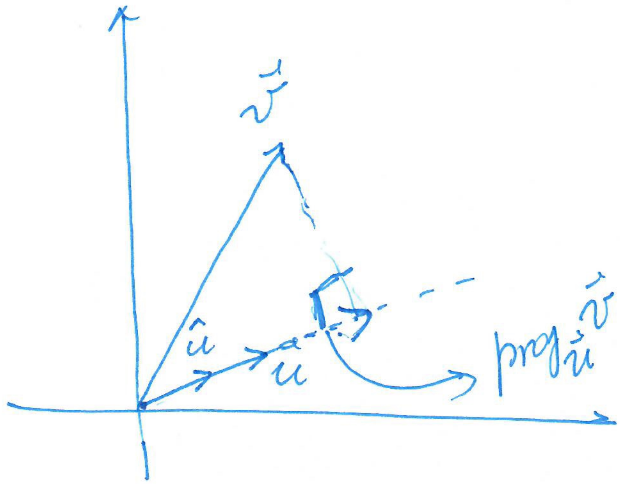
$$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) = (\vec{u} \cdot \vec{u}) + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + (\vec{v} \cdot \vec{v})$$

$$= \|\vec{u}\|^2 + 2(\vec{u} \cdot \vec{v}) + \|\vec{v}\|^2$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2$$

if and only if $(\vec{u} \cdot \vec{v}) = 0$.

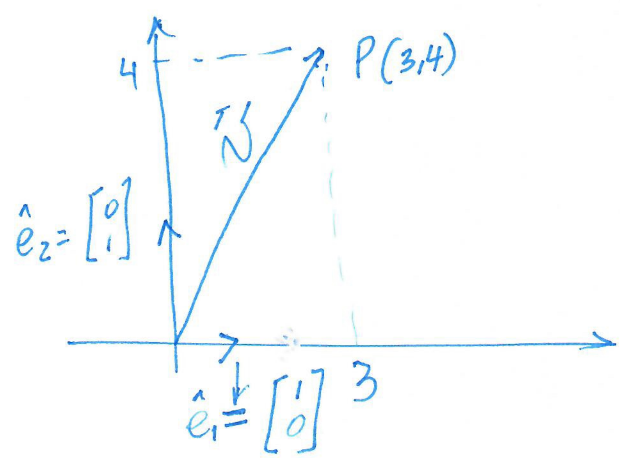
Projections



Proj. $\vec{v}$. in the direction of u

$$\boxed{\text{proj}_{\vec{u}}\vec{v} = (\vec{v}\cdot\hat{u})\,\hat{u}}, \quad \text{where } \hat{u} = \frac{\vec{u}}{\|\vec{u}\|} \qquad (9.1)$$

Motivation:
Consider $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$



$$\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$+ \left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = (\vec{v}\cdot\hat{e}_1)\hat{e}_1 + (v\cdot\hat{e}_2)\hat{e}_2.$$

$$\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$+ \left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= (\vec{v} \cdot \hat{e}_1)\hat{e}_1 + (\vec{v} \cdot \hat{e}_2)\hat{e}_2$$

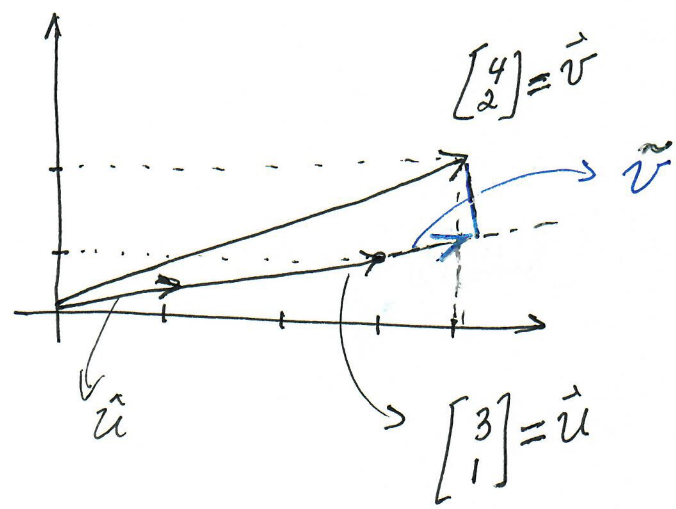
$$= \text{proj}_{\hat{e}_1} \vec{v} + \text{proj}_{\hat{e}_2} \vec{v}.$$

<u>Example:</u> Find $\text{proj}_{\vec{v}} \vec{u}$, when $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

Example: Given $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ in $\mathbb{R}^2$

Find the projection of $\vec{v}$ into $\vec{u}$.

or
find $\tilde{v}$, s.t., $\boxed{\tilde{v} = proj_{\vec{u}}\, \vec{v}}$



If $\hat{u} = \dfrac{\vec{u}}{\|\vec{u}\|}$ unit vector in direction of $\vec{u}$

According to formula (9.1)

$$\tilde{v} = proj_{\vec{u}}\, \vec{v} = (\vec{v} \cdot \hat{u})\hat{u}$$

In this particular case,

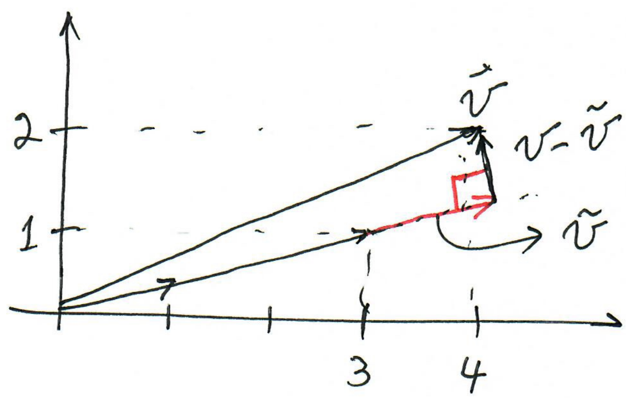
$\dfrac{\vec{u}}{\|\vec{u}\|} = \dfrac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, then.

$$\tilde{v} = (\vec{v}\cdot\hat{u})\hat{u} = \left(\begin{bmatrix}4\\2\end{bmatrix}\cdot\begin{bmatrix}3/\sqrt{10}\\1/\sqrt{10}\end{bmatrix}\right)\frac{1}{\sqrt{10}}\begin{bmatrix}3\\1\end{bmatrix}$$

$$= \frac{14}{\sqrt{10}}\left(\frac{1}{\sqrt{10}}\begin{bmatrix}3\\1\end{bmatrix}\right) = \frac{14}{10}\begin{bmatrix}3\\1\end{bmatrix} \text{ or } \boxed{\mathrm{Proj}_{\vec{u}}\vec{v} = \frac{14}{10}\begin{bmatrix}3\\1\end{bmatrix} = \frac{14}{10}\vec{u}}$$

So $\tilde{v}$ is a little larger than $\vec{u}$.

Now, I will show that $\boxed{\tilde{v}\perp(\vec{v}-\tilde{v})}$.



Proof.-

$$\frac{14}{10}\begin{bmatrix}3\\1\end{bmatrix}\cdot\left(\begin{bmatrix}4\\2\end{bmatrix}-\frac{14}{10}\begin{bmatrix}3\\1\end{bmatrix}\right) =$$

$$\frac{14}{10}\left(\begin{bmatrix}3\\1\end{bmatrix}\cdot\begin{bmatrix}4\\2\end{bmatrix}\right) - \left(\frac{14}{10}\right)^2\left(\begin{bmatrix}3\\1\end{bmatrix}\cdot\begin{bmatrix}3\\1\end{bmatrix}\right) = \frac{14}{10}(14) - \left(\frac{14}{10}\right)^2(10)$$

$$= 0.$$

In General,

<u>Def.</u> If $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ and $\vec{u} \neq \vec{0}$, then "the projection of $\vec{v}$ onto $\vec{u}$" is a new vector given by

$$\boxed{\operatorname{proj}_{\vec{u}} \vec{v} = (\vec{v} \cdot \hat{u})\,\hat{u}} \qquad (11.1)$$

Where $\hat{u} = \dfrac{\vec{u}}{\|\vec{u}\|}$

Therefore, $$\boxed{\operatorname{proj}_{\vec{u}} \vec{v} = \left(\vec{v} \cdot \frac{\vec{u}}{\|\vec{u}\|}\right) \frac{\vec{u}}{\|\vec{u}\|}} \qquad (11.2)$$

or $$\operatorname{proj}_{\vec{u}} \vec{v} = \left(\frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2}\right) \vec{u}$$

or $$\boxed{\operatorname{proj}_{\vec{u}} \vec{v} = \left(\frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\right) \vec{u}}. \qquad (11.3)$$